IMPROVEMENT OF MESH QUALITY USING A STATISTICAL APPROACH

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Abstract

This paper presents work in progress: a novel algorithm based on an adaptation of statistics that resulted in the prediction of a mesh quality criteria parameter. Experimental results are included to demonstrate the functionality of our method.

Key Words
Triangle mesh, statistics, mesh quality criteria, finite element method

1. Introduction

There are many optimization techniques for optimal point placement (see [1] and references therein). Most of them are based on the idea of local optimization and require an improvement of such mesh quality parameters as aspect ratio, area, and etc. Global mesh optimization has been studied recently in [2]. In spite of the well-known fact that local enhancement does not often provide reliable results, local methods are preferable in many applications dealing with rather large set of polygons, because the computational time involved is much less than for global-based techniques. In fact, these methods can be called deterministic methods. Recently, a statistical approach to the aspect ratio control was discussed in [3]. The main problem with this approach is that it is not clear how to determine a valid triangulation after a point placement according to the new predicted value of the aspect ratio. In fact, the method is presented for improving mesh quality, but without preserving the mesh's topology. For practical applications, it is necessary that results of a mesh optimization should allow the construction of topologically correct meshes. Moreover, it was noticed that in comparison with deterministic methods the approach [3], while offering some benefits, does not actually provide a better result. In practice, however it is very difficult or even impossible to generate a mesh with elements, whose aspect ratio values correspond exactly to predicted values. To attain good quality, some criteria on a mesh topology should be found and satisfied that is challenging itself.

In finite element analysis, the quality of the surface polygonization is also important (see, for instance, [4]). Let us here overview briefly methods more related to CAD applications.

There are two main ways of improving mesh quality: 1) modification of the mesh topology by inserting or deleting nodes, or by local reconnection; however, sometimes it is necessary to minimize changes in the topology of a surface; 2) methods that improve mesh quality without any topological changes. This approach is called smoothing.

Many smoothing techniques have been developed. Among the earliest of these methods is Laplacian smoothing [5] and its variations. The angle-based approach was proposed in [6]. In optimization-based methods, instead of moving each node on the basis of some geometric characteristic as is done in Laplacian and angle-based smoothing, so-called cost function is minimized. There are several kinds of cost functions: minimum / maximum angle [7,8], aspect ratio [9] and distortion metrics [10,11,12]. While optimization-based methods are very effective to avoid invalid elements, the computational time involved is much higher than for Laplacian and angle-based techniques. Therefore in [7,10] it was recommended that a more combined Laplacian / optimization-based approach should be applied. In physically based methods it is assumed that nodes are moved under the influence of some forces so that the shape of incident elements is improved. Lohner et al. [13] and Babuska et al. [14] consider a mesh as a system of springs that exert repulsive or attractive forces. In the methods of Shimada et al. [15, 16] and Bossen et al. [17] nodes are moved to attain equilibrium as centers of bubbles. Some physically based methods can be classified as optimization-based techniques. For instance, Djidjev [18] considered forces at each node. Simulation of these forces and the minimum energy configuration are defined by using an optimization procedure. In the recent years, CG community has paid more attention to mesh smoothing based on a signal processing approach, pioneered by Taubin in 1995 [19]. Rational filters offer an approach to mesh smoothing [20], where a Laplacian operator and a curvature operator are used to remove undesirable noise. Subdivision schemes are an alternative approach to the problem. Nevertheless, subdivision schemes [21] are able to deal with arbitrary topology but not with arbitrary connectivity, as was mentioned by Kobbelt et al. in [22]. Despite the many practical applications of triangle meshes in CG, there are applications such as FEM where well-shaped triangulations are needed; in addition to the deterioration in the accuracy of calculations, speed may be sacrificed in some applications. Still, well shaped triangulations may in fact be useful, as was mentioned in [23] "forty-odd years after the invention of the finite element method, our understanding of the relationship between mesh geometry, numerical accuracy, and stiffness matrix conditioning remains incomplete, even in the simplest cases." If a mesh is created for finite element method (FEM) applications, it is very important to control the mesh gradation smoothness. Shape elements have a strong influence on a discretization error. When the mesh gradations are not smooth, the discretization error of the neighboring elements changes markedly. From the physical point of view, this means that the relative stiffness of neighboring elements also changes significantly, which may cause problems in many tasks. For instance, all transient simulations are concerned with tracking phenomenon that propagates through the domain. If stiffness changes are too large propagation information can be made to incorrectly reflect back in the opposite direction, just as would occur when a stress wave hits the interface of materials of different stiffness [24]. Our main intention is to attain a fairly smooth change from one mesh element to other without considerable difference between the shapes of neighboring elements. Thus, our main premise in creating a mesh refinement algorithm based on statistical approach is that a more uniform (homogeneous) mesh in the sense of histogram distribution of elements in a histogram is more amenable to numerical calculations.

The rest of the paper is organized as follows. The algorithm of the method is discussed in Sec.2. Sec.3 contains conclusion remarks.

2. Algorithm

In presented work we suggest a novel algorithm based on an adaptation of statistics that results in the prediction of a mesh quality criteria according to the parameter values of its initial distribution. The main underlying assumption of the algorithm is that a local mesh refinement automatically results in improved global mesh quality, bearing in mind the distribution of a limited set of polygons in the entire model. The most significant difference from the previous approach [3] is that we do not generate the mesh exactly according to a new predicted value of the aspect ratio but instead use statistics on the values of the mesh quality criteria parameters of the neighbors of each vertex of the triangle mesh in order to predict the most likely value. This provides some latitude in the choice of point placement allowing softer "transformations" of polygons to be produced. For this reason we call it a Gentle Enhancement Method (GEM). Statistics deals with real numbers which correspond to individual elements of the polygonal mesh. These numbers are recognized in GEM as values of mesh quality criteria parameters. In our current work the mesh quality criteria parameter is calculated in 2D space, where 3D coordinates of local vertices are projected onto local average plane (see [25]) and defined as a ratio of the maximum edge length to the minimum edge length of an element of the triangle mesh (*m/mc*). This definition ensures a range of *m/mc* values from 1 to the maximum value. This range, denoted here as $X = [1, \max]$, is called probabilistic space. Usually, in statistics with a large amount of random sampling, experimental data are grouped into special intervals showing the number of elements that lie in such intervals of probabilistic space fragmentation. Since the empirical distribution of experimental data is discrete, we consider the frequency histogram as an analogue of density function over this space X . Considering the initial distribution of *m/mc* values, we assume that after refinement the distribution varies from a rather random distribution to a smoother one, such as a Gaussian normal distribution. A special mapping of an empirical distribution function to the desired smooth distribution (see Figure. 1) transforms *m/mc* values from the initial frequency histogram to a potential distribution (actually, one can require various distribution functions). This choice of distribution affects the new *m/mc* values predicted by probabilistic formulation.

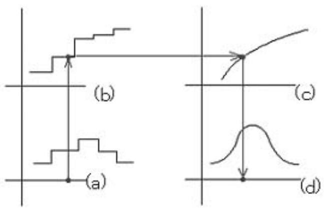


Figure 1. Mapping of a distribution function: (a) initial density; (b) initial distribution; (c) predicted distribution; (d) predicted density.

A technique is applied for an oriented pyramid: a star polygon, as it shown in Figure 2, centered at a processed point (it is called in the paper a star). The new center of a star with respect to its neighbors is calculated for each star by statistical means. We can consider a triangle element in the star with a fixed boundary and let the inner point of the star freely slide. Afterwards, for each element we find a new center that provides a potential shape of the element. This means that the element can have *m/mc* value found by prediction. These values are the ideal ones with respect to the star's neighbors. Finally, new center coordinates are the average values of potential center coordinates. Statistical parameters and characteristics of the global mesh quality shown below demonstrate the applicability of the proposed method. In the algorithm described, for each star with n elements and its *m/mc* values equal $a_1,...,a_n$ one can apply probabilistic analysis over intervals $\lfloor n_j, n_{j+1} \rfloor$ (for some integer number k, where $n_0 = a_{\min}$, $n_k = a_{\max}$) to calculate the distribution (histogram) in a current star and calculate new values $a_1^*,...,a_n^*$ predicted by formula (1) (see [3])

$$a_i^* = \sqrt{-2D\log(1-F_j)} + 1 \quad , \qquad (1)$$

where D is the variance or deviation.

Statistical parameters such as the deviation D of *m/mc* values from the average value M in the star, a probability F_j for the *m/mc* values a_i lying in interval $\lfloor n_0, n_{j+1} \rfloor$ can be found as follows:

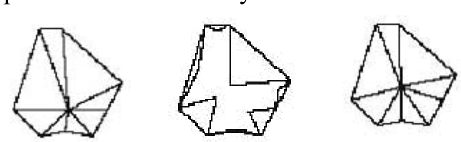
$$M = \frac{\sum_{j=1}^{n} a_j}{n} \;, \quad D = \frac{\sum_{j=1}^{n} (a_j - M)^2}{n} \;, \qquad (2)$$

$$f_j = \{ i : n_j < a_i \le n_{j+1} \} \;, \qquad (3)$$

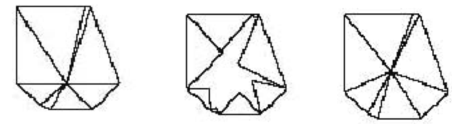
$$F_j = \frac{\{ i : a_i \le n_{j+1} \}}{n+1} \;, \qquad (4)$$

where f_j is a histogram value.

Figure 2 shows, from left to right, an initial star, potential triangles in the star, and averaged coordinates of inner potential points for an arbitrarily chosen star.



(a)

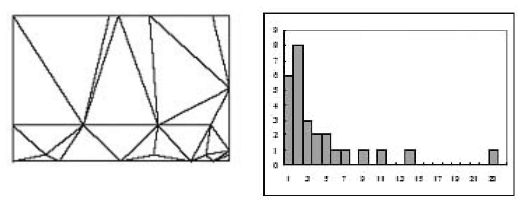


(b)

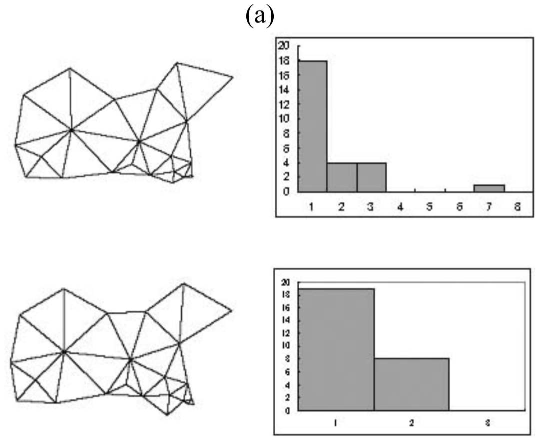
(c)

Figure 2. (a) An eight-triangle star. (b) An eight-triangle star with two highly degenerate triangles. (c) A two-triangle star.

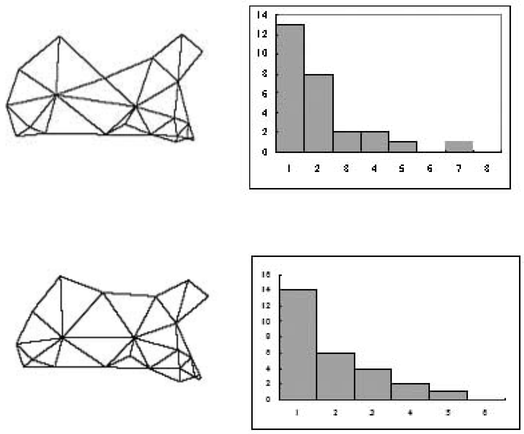
After processing an entire fragment of the model in accordance with the procedure described above, histogram values f_j are calculated for the fragment of a model. Figure 3 shows three examples that we investigated: an initial fragment, the improved fragment obtained by averaging according to equilateral triangles ($a_i^* = 1$) in each star, and the fragment after GEM application. From the histograms below we can see that although the average value M is better in the case (b) than in the case (c), the distribution of *m/mc* values approximates a Gaussian normal distribution. This means that GEM allows us to realize a gentle mesh improvement and to provide a rather smooth correction of the histogram.



(a)



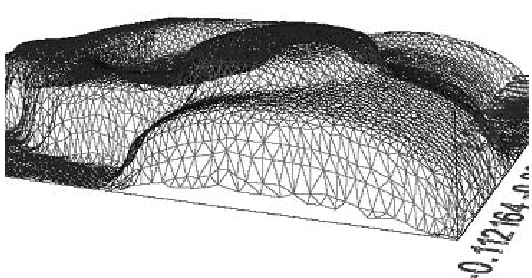
(b)



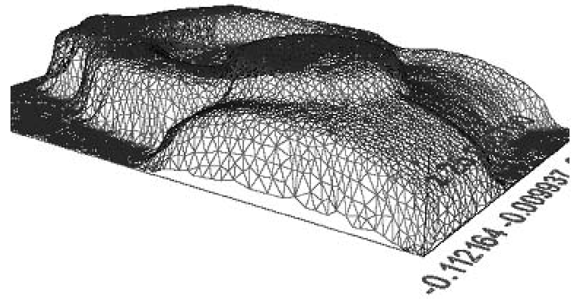
(c)

Figure 3. Mesh fragments and frequency histograms (category (x) axis – dynamic intervals of the *m/mc* value range X =[1.0, max]; value (y) axis – number of the mesh elements): (a) the initial (M = 2.84, D = 5.76), (b) according to $a_i^* = 1$ (1 step M = 1.45, D = 0.34; 2 steps M = 1.34, D = 0.06), (c) use of GEM (1 step M = 1.78, D = 0.51; 2 steps M = 1.68, D = 0.37).

As it was mentioned in Sec.1, our main premise in creating a mesh refinement algorithm based on statistical approach is that a more uniform (homogeneous) mesh in the sense of histogram distribution of elements is more amenable to numerical calculations. Figure 4 shows an example of the GEM implementation, which demonstrates that the shape of the object is changed slightly. The difference in volume between the initial (31234 triangles) mesh and final one (see Figure 4(b)) is about 0.6%. A visual inspection of the mesh shows that it looks more regular after processing (see also Figure 5). In Figure 6, we use the fragment of the initial mesh generated for the technical data set to demonstrate an implementation of a combination of polygon reduction and GEM.

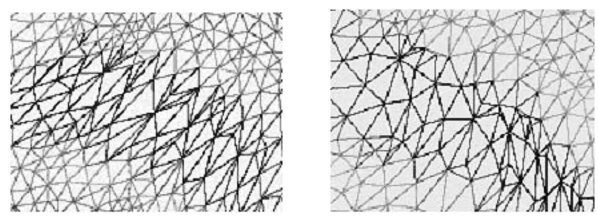


(a)



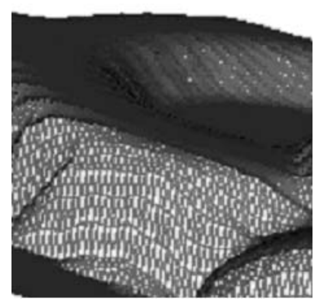
(b)

Figure 4. (a) Fragment of the mesh after polygon reduction (13967 triangles) and before use of GEM. (b) Fragment of the mesh after polygon reduction (13967 triangles) and after use of GEM. Processing time 3.615 sec (Intel Pentium III, 1000MHz/128MB RAM)

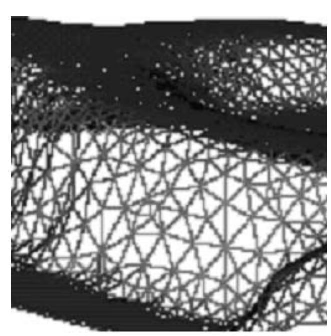


(a) (b)

Figure 5. Mesh fragment: (a) before processing; (b) after processing.



(a)



(b)

Figure 6. (a) Fragment of the initial mesh (31234 triangles). (b) Combined mesh modification (polygon reduction and GEM, 12132 triangles).

3. Conclusion

In the paper we proposed a novel statistical approach for triangle mesh improvement.
Future plans call for an extension of the improvement algorithm to exploit clusters of polygons (triangles, quadrilaterals) instead of the star made up triangles. We are currently investigating how to predict of *m/mc* values based on statistics to extend the plane method to 3D case. This can be achieved by improving a point placement according to an extrapolation and by using a mesh quality criteria parameter directly in 3D space. A compromise between this algorithm and the polygon reduction algorithm (see Figure 6) looks a very promising direction for further research.

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